J Math Chem (2012) 50:2317–2337
DOI 10.1007/s10910-012-0036-4

ORIGINAL PAPER

Applications of Haar basis method for solving some ill-posed inverse problems

R. Pourgholi · N. Tavallaie · S. Foadian

Received: 21 November 2011 / Accepted: 28 May 2012 / Published online: 6 June 2012
© Springer Science+Business Media, LLC 2012

Abstract In this paper a numerical method consists of combining Haar basis method and Tikhonov regularization method for solving some ill-posed inverse problems using noisy data is presented. By using a sensor located at a point inside the body and measuring the $u(x,t)$ at a point $x = a$, $0 < a < 1$, and applying Haar basis method to the inverse problem, we determine a stable numerical solution to this problem. Results show that an excellent estimation on the unknown functions of the inverse problem can be obtained within a couple of minutes CPU time at pentium IV-2.4 GHz PC.

Keywords Ill-posed inverse problems · Haar basis method · Tikhonov regularization method · Noisy data

Mathematics Subject Classification (2000) 65M32 · 35K05

1 Introduction

Inverse problems appear in many important scientific and technological fields. Hence analysis, design implementation and testing of inverse algorithms are also are great scientific and technological interest.

R. Pourgholi (✉) · N. Tavallaie · S. Foadian
School of Mathematics and Computer Sciences, Damghan University, P.O. Box 36715-364, Damghan, Iran
e-mail: pourgholi@du.ac.ir

N. Tavallaie
e-mail: tavallaie@du.ac.ir

S. Foadian
e-mail: sfo.1365@gmail.com

Mathematically, the inverse problems belong to the class of problems called the ill-posed problems, i.e. small errors in the measured data can lead to large deviations in the estimated quantities. As a consequence, their solution does not satisfy the general requirement of existence, uniqueness, and stability under small changes to the input data. To overcome such difficulties, a variety of techniques for solving inverse problems have been proposed [1–6,14,15,19,20,22–26,30] and among the most versatile methods the following can be mentioned: Tikhonov regularization [28], iterative regularization [2], mollification [20], BFM (Base Function Method) [23], SFDM (Semi Finite Difference Method) [19], and the FSM (Function Specification Method) [3].

Beck and Murio [5] presented a new method that combines the function specification method of Beck with the regularization technique of Tikhonov. Murio and Paloschi [21] propose a combined procedure based on a data filtering interpretation of the mollification method and FSM. Beck et al. [3] compare the FSM, the Tikhonov regularization and the iterative regularization, using experimental data.

Zhou et al. [30] investigated the inverse heat conduction problem in a one-dimensional composite slab with rate-dependent pyrolysis chemical reaction and outgassing flow effects using the iterative regularization approach. They considered the thermal properties of the temperature-dependent composites.

Huanga et al. [15] applied an iterative regularization method based inverse algorithm in the present study in simultaneously determining the unknown temperature and concentration-dependent heat and mass production rates for a chemically reacting fluid by using interior measurements of temperature and concentration.

Haar functions, [12], have been used from 1910 when they were introduced by the Hungarian mathematician Haar [11]. The Haar transform is one of the earliest of what is known now as a compact, dyadic, orthonormal wavelet transform. The Haar function, being an odd rectangular pulse pair, is the simplest and oldest orthonormal wavelet with compact support. In the mean time, several definitions of the Haar functions and various generalizations have been published and used. They were intended to adopt this concept to some practical applications as well as to extend its in applications to different classes of signals. Haar functions appear very attractive in many applications as for example, image coding, edge extraction and binary logic design.

Recently, Haar wavelets, [12], have been applied extensively for signal processing in communications and physics research, and have proved to be a wonderful mathematical tool. After discretizing the differential equations in a conventional way like the finite difference approximation, wavelets can be used for algebraic manipulations in the system of equations obtained which lead to better condition number of the resulting system.

The previous work, [12], in the system analysis via Haar wavelets was led by Chen and Hsiao [7], who first derived a Haar operational matrix for the integrals of the Haar functions vector and put the application for the Haar analysis into the dynamical systems. Then, the pioneer work in state analysis of linear time delayed systems via Haar wavelets was laid down by Hsiao [13], who first proposed a Haar product matrix and a coefficient matrix. Hsiao and Wang proposed a key idea to transform the time-varying function and its product with states into a Haar product matrix. Kalpana and Balachandar [16] presented Haar basis method of analysis for observer design in the generalized state space or singular system of transistor circuits.

The plan of this paper is as follows. In Sect. 2, we formulate and solve an inverse problem for the heat equation. In addition, a solution of an inverse problem for the wave equation will be discussed. To regularize the resultant ill-conditioned linear system of equations, we apply the Tikhonov regularization (of 0th, 1st and 2nd orders) method to obtain the stable numerical approximation of our solution. Finally some numerical results will be given in Sect. 3.

2 Main results

Definition 2.1 The Haar wavelet family for $x \in [0, 1)$ is defined as follows, [12],

$$h_i(x) = \begin{cases} 1, & x \in \left[\frac{k}{m}, \frac{k+0.5}{m}\right), \\ -1, & x \in \left[\frac{k+0.5}{m}, \frac{k+1}{m}\right), \\ 0, & \text{elsewhere.} \end{cases}$$

Integer $m = 2^j$, $(j = 0, 1, \ldots, J)$ indicates the level of the wavelet; $k = 0, 1, \ldots, m-1$ is the translation parameter. Maximal level of resolution is J. The index i is calculated by $i = m + k + 1$; in the case of minimal values $m = 1, k = 0$ we have $i = 2$, the maximal value of i is $i = 2^{J+1} = M$. It is assumed that the value $i = 1$ corresponds to the scaling function for which $h_1 \equiv 1$ in $[0, 1)$. Let us define the collocation point $x_l = \frac{l-0.5}{M}$, $(l = 1, 2, \ldots, M)$ and discretize the Haar functions $h_i(x)$. In this way we get the coefficient matrix H and the operational matrices of integration P and Q, which are M-square matrices, are defined by the equations

$$(H)_{il} = (h_i(x_l)), \tag{2.1}$$

$$(PH)_{il} = \int_0^{x_l} h_i(x)\, dx, \tag{2.2}$$

$$(QH)_{il} = \int_0^{x_l}\int_0^{x} h_i(s)\, ds\, dx. \tag{2.3}$$

The elements of the matrices H, P and Q can be evaluated by (2.1), (2.2) and (2.3). For example when $M = 2, 4$ we have,

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P_2 = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \quad Q_2 = \frac{1}{32}\begin{pmatrix} 5 & -4 \\ 4 & -3 \end{pmatrix},$$

$$H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad P_4 = \frac{1}{16}\begin{pmatrix} 8 & -4 & -2 & -2 \\ 4 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix},$$

$$Q_4 = \frac{1}{128}\begin{pmatrix} 21 & -16 & -4 & -12 \\ 16 & -11 & -4 & -4 \\ 6 & -2 & -3 & 0 \\ 2 & -2 & 0 & -3 \end{pmatrix}.$$

Remark 1 Any function $f \in L^2([0, 1))$ can be decomposed as, [12],

$$f(x) = \sum_{n=1}^{\infty} c_n h_n(x), \tag{2.4}$$

where the coefficients c_n are determined by

$$c_n = 2^j \int_0^1 f(x) h_n(x)\, dx,$$

where $n = 2^j + k,\ j \geqslant 0,\ 0 \leqslant k < 2^j$.

We should note by Remark 1 that if $f(x)$ is piecewise constant by itself, or may be approximated as piecewise constant during each subinterval, then $f(x)$ will be terminated at finite terms, that is,

$$f(x) = \sum_{n=1}^{M} c_n h_n(x) = C_M^T H_M(x), \tag{2.5}$$

where the coefficients C_M^T and the Haar function vector $H_M(x)$ are defined as,

$$H_M(x) = \left(h_1(x)\ h_2(x)\ \ldots\ h_M(x)\right)^T,$$
$$C_M^T = \left(c_1\ c_2\ \ldots\ c_M\right),$$

where 'T' means transpose and $M = 2^{J+1}$.

2.1 Inverse problem for the heat equation

One example of the inverse heat conduction problem is the estimation of the heating history experienced by a shuttle or missile reentering the earth's atmosphere from space. The heat flux at the heated surface is needed [3]. To estimate the surface heat flux history, it is necessary to have a mathematical model of the heat transfer process. For example, it is assumed that the section of the skin is of a single material, homogeneous and isotropic, and that it closely approximates a flat plate. Then a possible mathematical model for the temperature in the plate is a one dimensional inverse heat conduction problem as follows, [3]:

$$u_t(x,t) = u_{xx}(x,t), \qquad 0 < x < 1, \quad 0 < t < t_f, \tag{2.6a}$$

$$u(x,0) = \phi(x), \qquad 0 \le x \le 1, \tag{2.6b}$$

$$u(0,t) = p(t), \qquad 0 \le t \le t_f, \tag{2.6c}$$

$$u(1,t) = q(t), \qquad 0 \le t \le t_f, \tag{2.6d}$$

and the overspecified condition

$$u(a,t) = g(t), \quad 0 \le t \le t_f, \tag{2.6e}$$

where $0 < a < 1$ is a fixed point, $\phi(x)$ is a continuous known function, $g(t)$ and $q(t)$ are infinitely differentiable known functions and t_f represents the final time, while the function $p(t)$ is unknown which remains to be determined from some interior temperature measurements.

Now, let us divide the interval $[0, t_f]$ into N equal parts of length $\Delta t = \frac{t_f}{N}$ and denote $t_s = (s-1)\Delta t,\ s = 1, 2, \ldots, N$. We assume that $\dot{u}''$ can be expanded in terms of Haar wavelets as, [12]

$$\dot{u}''(x,t) = \sum_{n=1}^{M} c_s(n) h_n(x) = C_M^T H_M(x), \tag{2.7}$$

where $\cdot$ and $'$ mean differentiation with respect to t and x, respectively, the vector C_M^T is constant in each subinterval $[t_s, t_{s+1}],\ s = 1, 2, \ldots, N$.

Integrating formula (2.7) with respect to t from t_s to t and then twice with respect to x from a to x, we obtain

$$u''(x,t) = (t - t_s) C_M^T H_M(x) + u''(x, t_s), \tag{2.8}$$

$$\begin{aligned} u(x,t) = {} & (t - t_s) C_M^T [Q_M H_M(x) - Q_M H_M(a) - (x-a) P_M H_M(a)] \\ & + u(a,t) - u(a,t_s) + (x-a)[u'(a,t) - u'(a,t_s))] + u(x,t_s), \end{aligned} \tag{2.9}$$

$$\begin{aligned} \dot{u}(x,t) = {} & C_M^T [Q_M H_M(x) - Q_M H_M(a) - (x-a) P_M H_M(a)] \\ & + \dot{u}(a,t) + (x-a)\dot{u}'(a,t). \end{aligned} \tag{2.10}$$

By using the boundary conditions, we obtain

$$u(a,t_s) = g(t_s), \ \ u(1,t_s) = q(t_s), \ \ \dot{u}(a,t) = g'(t), \ \ \dot{u}(1,t) = q'(t).$$

Putting $x = 1$ in (2.9) and (2.10), we obtain

$$\begin{aligned} u'(a,t) - u'(a,t_s) = {} & \frac{t - t_s}{a-1} C_M^T [Q_M H_M(1) - Q_M H_M(a) - (1-a) P_M H_M(a)] \\ & + \frac{1}{1-a}(u(1,t) - u(1,t_s)) + \frac{1}{1-a}(u(a,t_s) - u(a,t)), \end{aligned} \tag{2.11}$$

$$\dot{u}'(a,t) = \frac{1}{1-a}[\dot{u}(1,t) - \dot{u}(a,t)] - C_M^T\left[\frac{1}{1-a}Q_M H_M(1)\right.$$
$$\left. -\frac{1}{1-a}Q_M H_M(a) - P_M H_M(a)\right]. \tag{2.12}$$

Substituting Eqs. (2.11) and (2.12) into Eqs. (2.9) and (2.10), and discretizing the results by assuming $x \to x_l$, $t \to t_{s+1}$, we obtain

$$u''(x_l, t_{s+1}) = (t_{s+1} - t_s)C_M^T H_M(x_l) + u''(x_l, t_s), \tag{2.13}$$

$$u(x_l, t_{s+1}) = (t_{s+1} - t_s)C_M^T\left[Q_M H_M(x_l) - \frac{x_l - a}{1-a}P_M F + \frac{x_l - 1}{1-a}Q_M H_M(a)\right]$$
$$+u(x_l, t_s) + \frac{x_l - 1}{1-a}[g(t_s) - g(t_{s+1})] + \frac{x_l - a}{1-a}[q(t_{s+1}) - q(t_s)], \tag{2.14}$$

$$\dot{u}(x_l, t_{s+1}) = C_M^T\left[Q_M H_M(x_l) - \frac{x_l - a}{1-a}P_M F + \frac{x_l - 1}{1-a}Q_M H_M(a)\right]$$
$$+\frac{1 - x_l}{1-a}[g'(t_{s+1})] + \frac{x_l - a}{1-a}[q'(t_{s+1})]. \tag{2.15}$$

where the vector F is defined as

$$F = [1, \underbrace{0, \ldots, 0}_{(M-1)}]^T$$

and H, P, Q are obtained from (2.1)–(2.3).

In the following scheme

$$\dot{u}(x_l, t_{s+1}) = u''(x_l, t_{s+1}), \tag{2.16}$$

which leads us from the time layer t_s to t_{s+1} is used where x_l is collocation point. Substituting (2.13) and (2.15) into (2.16), we obtain

$$C_M^T\left[Q_M H_M(x_l) - \frac{x_l - a}{1-a}P_M F + \frac{x_l - 1}{1-a}Q_M H_M(a) - \Delta t H_M(x_l)\right]$$
$$= u''(x_l, t_s) - \frac{1 - x_l}{1-a}g'(t_{s+1}) - \frac{x_l - a}{1-a}q'(t_{s+1}), \tag{2.17}$$

Thus the linear system corresponding to the wavelet coefficient C_M^T can be expressed as

$$\Lambda\Theta = B. \tag{2.18}$$

The Matrix Λ is ill-conditioned. On the other hand, as $g(t)$ is affected by measurement errors, the estimate of Θ by (2.18) will be unstable so that the Tikhonov regularization method must be used to control this measurement errors. The Tikhonov regularized solution ([10,17,27] and [28]) to the system of linear algebraic equation (2.18) is given by

$$F_\alpha(\Theta) = \|\Lambda\Theta - B\|_2^2 + \alpha\|R^{(s)}\Theta\|_2^2.$$

On the case of the zeroth-, first-, and second-order Tikhonov regularization method the matrix $R^{(s)}$, for $s = 0, 1, 2$, is given by, see e.g. [18]:

$$\begin{aligned}
R^{(0)} &= I_{M_1 \times M_1} \in \mathbb{R}^{M_1 \times M_1},\\
R^{(1)} &= \begin{pmatrix} -1 & 1 & \dots & 0 & 0 & 0\\ 0 & -1 & 1 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & -1 & 1 & 0\\ 0 & 0 & \dots & 0 & -1 & 1 \end{pmatrix} \in \mathbb{R}^{(M_1-1)\times M_1},\\
R^{(2)} &= \begin{pmatrix} 1 & -2 & 1 & 0 & \dots & 0 & 0\\ 0 & 1 & -2 & 1 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 1 & -2 & 1 & 0\\ 0 & 0 & \dots & 0 & 1 & -2 & 1 \end{pmatrix} \in \mathbb{R}^{(M_1-2)\times M_1},
\end{aligned}$$

where $M_1 = (\gamma + 1) \times (\iota + 1)$.

Therefore, we obtain the Tikhonov regularized solution of the regularized equation as

$$\Theta_\alpha = \left[\Lambda^T\Lambda + \alpha(R^{(s)})^T R^{(s)}\right]^{-1} \Lambda^T B.$$

In our computation, we use the GCV scheme to determine a suitable value of α ([8,9] and [29]).

2.2 Inverse problem for the wave equation

In this section, we consider the following inverse problem for the wave equation in the dimensionless form

$$u_{tt}(x,t) = u_{xx}(x,t), \qquad 0 < x < 1, \quad 0 < t < t_f, \tag{2.19a}$$
$$u(x,0) = f_1(x), \qquad 0 \le x \le 1, \tag{2.19b}$$
$$u_t(x,0) = f_2(x), \qquad 0 \le x \le 1, \tag{2.19c}$$
$$u(0,t) = p(t), \qquad 0 \le t \le t_f, \tag{2.19d}$$
$$u(1,t) = q(t), \qquad 0 \le t \le t_f, \tag{2.19e}$$

and the overspecified condition

$$u(a,t) = g(t), \quad 0 \le t \le t_f, \tag{2.19f}$$

where $0 < a < 1$ is a fixed point, $f_1(x)$ is a continuous known function, $g(t)$ and $q(t)$ are infinitely differentiable known functions and t_f represents the final time; while the function $p(t)$ is unknown which remains to be determined from some interior temperature measurements.

Now, let us divide the interval $[0, t_f]$ into N equal parts of length $\Delta t = \frac{t_f}{N}$ and denote $t_s = (s-1)\Delta t,\ s = 1, 2, \ldots, N$. We assume by Remark 1 that $\ddot{u}''$ can be expanded in terms of h_n-functions as,

$$\ddot{u}''(x,t) = \sum_{n=1}^{M} c_s(n) h_n(x) = C_M^T H_M(x), \tag{2.20}$$

where $\cdot = \partial/\partial t$ and $' = \partial/\partial x$ and the vector C_M^T is constant in each subinterval $[t_s, t_{s+1}],\ s = 1, 2, \ldots, N$.

Integrating formula (2.20) twice with respect to t from t_s to t and then twice with respect to x from a to x, we obtain

$$u''(x,t) = \frac{1}{2}(t^2 + t_s^2 - 2tt_s) C_M^T H_M(x) + u''(x,t_s) + (t-t_s)\dot{u}''(x,t_s), \tag{2.21}$$

$$\begin{aligned}\ddot{u}(x,t) = {} & C_M^T [Q_M H_M(x) - Q_M H_M(a) - (x-a) P_M H_M(a)] \\ & + \ddot{u}(a,t) + (x-a)\ddot{u}'(a,t),\end{aligned} \tag{2.22}$$

$$\begin{aligned}u(x,t) = {} & \frac{1}{2}(t^2 + t_s^2 - 2tt_s) C_M^T [Q_M H_M(x) - Q_M H_M(a) - (x-a) P_M H_M(a)] \\ & + u(x,t_s) + (t-t_s)\dot{u}(x,t_s) + u(a,t) - u(a,t_s) - (t-t_s)\dot{u}(a,t_s) \\ & + (x-a)[u'(a,t) - u'(a,t_s) - (t-t_s)\dot{u}'(a,t_s)].\end{aligned} \tag{2.23}$$

By using the boundary conditions, we obtain

$$\begin{aligned} u(1,t_s) &= q(t_s), \quad u(a,t_s) = g(t_s), \\ \dot{u}(1,t_s) &= q'(t_s), \quad \dot{u}(a,t_s) = g'(t_s), \\ \ddot{u}(1,t) &= q''(t), \quad \ddot{u}(a,t) = g''(t). \end{aligned}$$

Putting $x = 1$ in (2.22) and (2.23), we obtain

$$\ddot{u}'(a,t) = C_M^T\left[\frac{1}{a-1}Q_M H_M(1) - \frac{1}{a-1}Q_M H_M(a) + P_M H_M(a)\right] + \frac{1}{1-a}[q''(t) - g''(t)], \tag{2.24}$$

$$\begin{aligned} u'(a,t) - u'(a,t_s) - (t-t_s)\dot{u}'(a,t_s) &= \frac{1}{1-a}[q(t)-q(t_s)] \\ &+ \frac{1}{1-a}[g(t_s)-g(t)] + \frac{t-t_s}{1-a}[g'(t_s)-q'(t_s)] \\ &+ \frac{1}{2}(t^2+t_s^2-2tt_s)C_M^T\left[\frac{1}{a-1}Q_M H_M(1) - \frac{1}{a-1}Q_M H_M(a) + P_M H_M(a)\right]. \end{aligned} \tag{2.25}$$

Substituting (2.24) and (2.25) into (2.22) and (2.23), and discretizing the results by assuming $x \to x_l$, $t \to t_{s+1}$ we obtain

$$\begin{aligned} u''(x_l,t_{s+1}) &= \frac{1}{2}(t_{s+1}^2 + t_s^2 - 2t_{s+1}t_s)C_M^T H_M(x_l) \\ &+ u''(x_l,t_s) + (t_{s+1}-t_s)\dot{u}''(x_l,t_s), \end{aligned} \tag{2.26}$$

$$\dot{u}''(x_l,t_{s+1}) = (t_{s+1}-t_s)C_M^T H_M(x_l) + \dot{u}''(x_l,t_s), \tag{2.27}$$

$$\begin{aligned} \ddot{u}(x_l,t_{s+1}) &= C_M^T\left[Q_M H_M(x_l) + \frac{1-x_l}{a-1}Q_M H_M(a) + \frac{x_l-a}{a-1}P_M F\right] \\ &+ \frac{1-x_l}{1-a}g''(t_{s+1}) + \frac{x_l-a}{1-a}q''(t_{s+1}), \end{aligned} \tag{2.28}$$

$$\begin{aligned} u(x_l,t_{s+1}) &= \frac{1}{2}(t_{s+1}^2 + t_s^2 - 2t_{s+1}t_s)C_M^T\left[Q_M H_M(x_l) + \frac{1-x_l}{a-1}Q_M H_M(a)\right. \\ &\left. + \frac{x_l-a}{a-1}P_M F\right] + u(x_l,t_s) + (t_{s+1}-t_s)\dot{u}(x_l,t_s) \\ &+ \frac{x_l-1}{1-a}[g(t_s) - g(t_{s+1})] + \frac{x_l-a}{1-a}[q(t_{s+1}) - q(t_s)] \\ &+ \frac{t_{s+1}-t_s}{1-a}[(x_l-1)g'(t_s) - (x_l-a)q'(t_s)], \end{aligned} \tag{2.29}$$

$$\begin{aligned} \dot{u}(x_l,t_{s+1}) &= (t_{s+1}-t_s)C_M^T\left[Q_M H_M(x_l) + \frac{1-x_l}{a-1}Q_M H_M(a) + \frac{x_l-a}{a-1}P_M F\right] \\ &+ \dot{u}(x_l,t_s) + \frac{1-x_l}{1-a}[g'(t_{s+1}) - g'(t_s)] + \frac{x_l-a}{1-a}[q'(t_{s+1}) - q'(t_s)]. \end{aligned} \tag{2.30}$$

where the vector F is defined as

$$F = [1, \underbrace{0, \ldots, 0}_{(M-1)}]^T$$

and H, P and Q are obtained by (2.1), (2.2) and (2.3). In the following scheme

$$\ddot{u}(x_l, t_{s+1}) = u''(x_l, t_{s+1}), \tag{2.31}$$

which leads us from the time layer t_s to t_{s+1} is used where x_l is collocation point. Substituting Eqs. (2.26) and (2.28) into Eq. (2.31), we obtain

$$\begin{aligned} &C_M^T \left[Q_M H_M(x_l) + \frac{1-x_l}{a-1} Q_M H_M(a) + \frac{x_l - a}{a-1} P_M F \right. \\ &\left. -\frac{1}{2}(t_{s+1}^2 + t_s^2 - 2t_{s+1}t_s) H_M(x_l) \right] = u''(x_l, t_s) + \Delta t \dot{u}''(x_l, t_s) \\ &-\frac{1-x_l}{1-a} g''(t_{s+1}) - \frac{x_l - a}{1-a} q''(t_{s+1}). \end{aligned} \tag{2.32}$$

From the formula (2.32) the wavelet coefficient C_M^T can be calculated.

In matrix form, the wavelet coefficient C_M^T, can be obtained from solving the following matrix equation

$$A\lambda = b. \tag{2.33}$$

Similarly, the Tikhonov regularized solution to the system of linear algebraic equation (2.18) is given by (see e.g. [10,17] and [27])

$$\lambda_\alpha = \left[A^T A + \alpha (R^{(s)})^T R^{(s)} \right]^{-1} A^T b.$$

Table 1 The comparison between exact and Tikhonov solutions of $p(t)$ with noisy data

t	Exact	0th order Tikhonov	1st order Tikhonov	2nd order Tikhonov
0.01	1.921879	1.919898	1.922970	1.922109
0.02	1.847433	1.844146	1.848860	1.847618
0.1	1.370640	1.361024	1.370366	1.369286
0.11	1.324373	1.314032	1.323874	1.322876
0.5	1.020671	0.991988	1.017550	1.017308
0.51	1.040357	1.011336	1.037165	1.036948
0.8	2.001524	1.962028	1.998064	1.997799
0.81	2.046628	2.006747	2.043149	2.042928
0.9	2.484647	2.441109	2.481159	2.480956
0.91	2.536805	2.492833	2.533312	2.533111
1	3.036631	2.988876	3.033070	3.033044
	S	3.053e−002	2.875e−003	3.129e−003

Table 2 The comparison between exact and Tikhonov solutions of $u(0.2, t)$ with noisy data

t	Exact	0th order Tikhonov	1st order Tikhonov	2nd order Tikhonov
0.01	2.520089	2.521307	2.519971	2.520238
0.02	2.423450	2.425538	2.423526	2.423825
0.1	1.799281	1.806399	1.801775	1.801910
0.11	1.737893	1.745568	1.740668	1.740750
0.5	1.165108	1.186168	1.171230	1.171247
0.51	1.182700	1.203986	1.188859	1.188836
0.8	2.123236	2.151417	2.129903	2.129798
0.81	2.168547	2.196991	2.175235	2.175103
0.9	2.610014	2.640830	2.616710	2.616683
0.91	2.662706	2.693775	2.669420	2.669383
1	3.168405	3.201933	3.175197	3.175122
	S	2.199e−002	5.703e−003	5.652e−003

Table 3 The comparison between exact and Tikhonov solutions of $u(x, 0.5)$ with noisy data

t	Exact	0th order Tikhonov	1st order Tikhonov	2nd order Tikhonov
0	1.020671	0.991984	1.017502	1.017258
0.1	1.084074	1.084074	1.084074	1.084074
0.2	1.165108	1.186201	1.171323	1.171244
0.3	1.263251	1.291492	1.271317	1.270897
0.4	1.379145	1.411504	1.389255	1.388597
0.5	1.514630	1.549149	1.526990	1.526255
0.6	1.672755	1.704997	1.684498	1.683785
0.7	1.857762	1.884610	1.866750	1.866089
0.8	2.075052	2.098842	2.084764	2.084199
0.9	2.331120	2.342195	2.334015	2.333667
1	2.633481	2.633481	2.633481	2.633481
	S	2.598e−002	8.307e−003	7.831e−003

3 Numerical results and discussion

In this section, we are going to study numerically the inverse problems (2.6) and (2.19) with the unknown boundary condition. The main aim here is to show the applicability of the present method, described in Sect. 2, for solving the inverse problems (2.6) and (2.19). As expected the inverse problems are ill-posed and therefore it is necessary to investigate the stability of the present method by giving a test problem.

Remark 2 In an inverse problem there are two sources of error in the estimation; the first source is the unavoidable bias deviation, and the second source of error is the variance due to the amplification of measurement errors, [6].

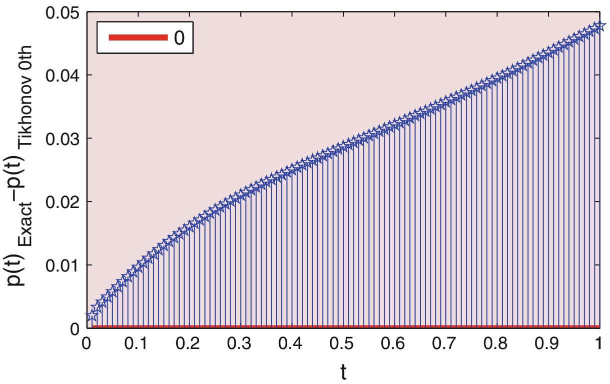

Fig. 1 Difference between the $p(t)_{\text{Exact}}$ and $p(t)_{\text{0th order Tikhonov}}$ of problem (2.6) with noisy data

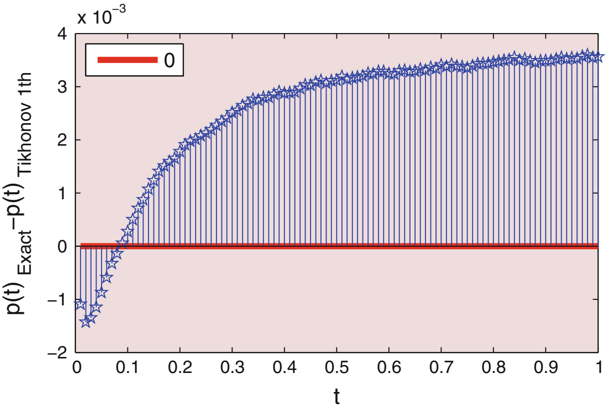

Fig. 2 Difference between the $p(t)_{\text{Exact}}$ and $p(t)_{\text{1st order Tikhonov}}$ of problem (2.6) with noisy data

Therefore, we compare exact and approximate solutions by considering total error S defined by

$$S = \left[\frac{1}{N-1} \sum_{i=1}^{N} (\widehat{\Phi_i} - \Phi_i)^2 \right]^{\frac{1}{2}}, \tag{3.1}$$

where N, Φ and $\widehat{\Phi}$ are the number of estimated values, the estimated values and the exact values, respectively.

Example 3.1 In this example we solve the problem (2.6) with given data,

$$u(x,0) = 2(\sin(2x) + \cos(2x)) + \frac{1}{4}x^4, \quad 0 \le x \le 1,$$
$$u(1,t) = 2e^{-4t}(\sin 2 + \cos 2) + 3\left(t^2 + t + \frac{1}{12}\right), \quad 0 \le t \le t_f,$$

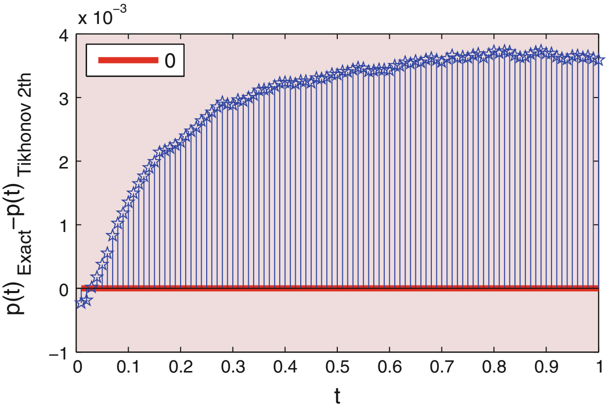

Fig. 3 Difference between the $p(t)_{\text{Exact}}$ and $p(t)_{\text{2nd order Tikhonov}}$ of problem (2.6) with noisy data

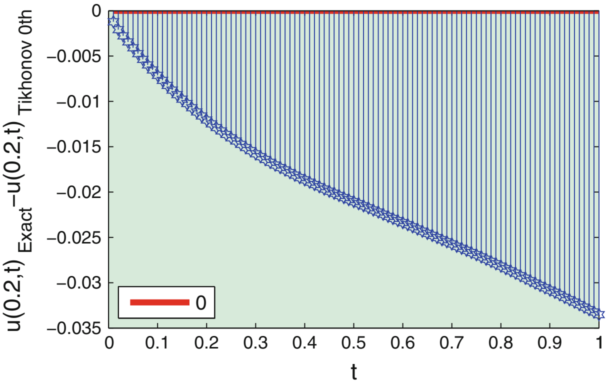

Fig. 4 Difference between the $u(0.2, t)_{\text{Exact}}$ and $u(0.2, t)_{\text{0th order Tikhonov}}$ of problem (2.6) with noisy data

$$u(0.1, t) = 2e^{-4t}(\sin 0.2 + \cos 0.2) + 3\left(t^2 + (0.01)t + \frac{0.0001}{12}\right), \quad 0 \le t \le t_f.$$

The exact solution of this problem is

$$u(x, t) = 2e^{-4t}(\sin(2x) + \cos(2x)) + 3\left(t^2 + tx^2 + \frac{1}{12}x^4\right).$$

Our results obtained for $p(t) = u(0, t)$, $u(0.2, t)$ and $u(x, 0.5)$ when $t_f = 1$, $\Delta t = 0.01$ and $\Delta x = \frac{1}{4}$ with noisy data (noisy data = input data + (0.01)rand(1)) are presented in Tables 1, 2 and 3 and Figs. 1, 2, 3, 4, 5, 6, 7, 8 and 9.

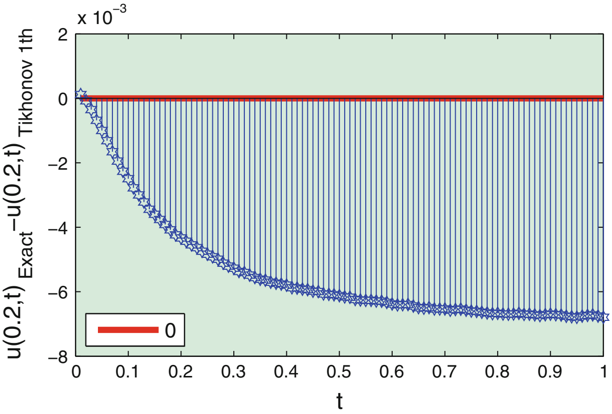

Fig. 5 Difference between the $u(0.2, t)_{\text{Exact}}$ and $u(0.2, t)_{\text{1st order Tikhonov}}$ of problem (2.6) with noisy data

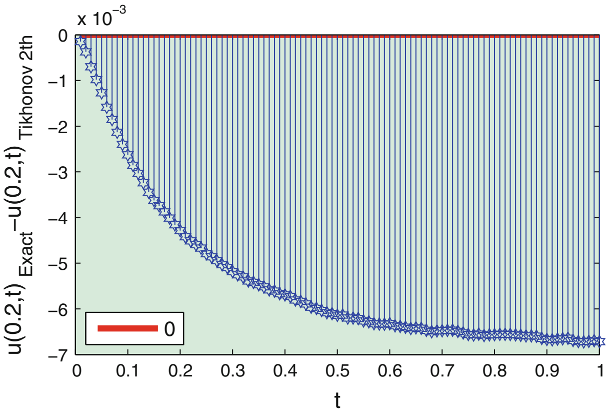

Fig. 6 Difference between the $u(0.2, t)_{\text{Exact}}$ and $u(0.2, t)_{\text{2nd order Tikhonov}}$ of problem (2.6) with noisy data

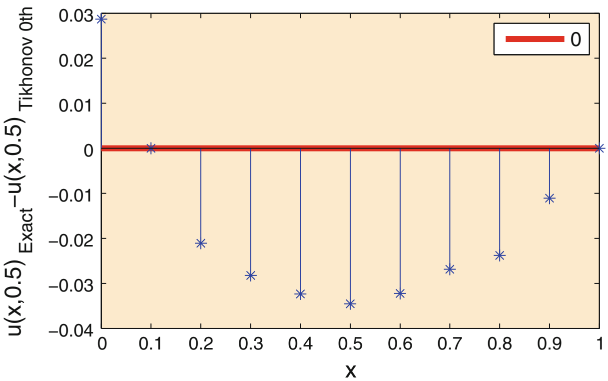

Fig. 7 Difference between the $u(x, 0.5)_{\text{Exact}}$ and $u(x, 0.5)_{\text{0th order Tikhonov}}$ of problem (2.6) with noisy data

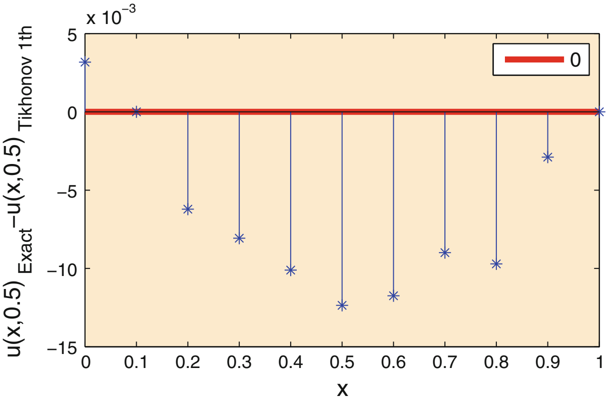

Fig. 8 Difference between the $u(x, 0.5)_{\text{Exact}}$ and $u(x, 0.5)_{\text{1st order Tikhonov}}$ of problem (2.6) with noisy data

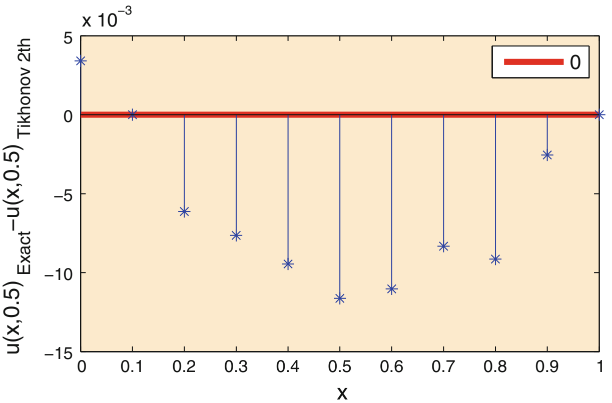

Fig. 9 Difference between the $u(x, 0.5)_{\text{Exact}}$ and $u(x, 0.5)_{\text{2nd order Tikhonov}}$ of problem (2.6) with noisy data

Example 3.2 In this example we solve the problem (2.19) with given data,

$$\begin{aligned} u(x, 0) &= e^{-x} + x^2, \quad 0 \le x \le 1, \\ u_t(x, 0) &= -e^{-x} + x^3, \quad 0 \le x \le 1, \\ u(1, t) &= e^{-1-t} + t + t^3 + t^2 + 1, \quad 0 \le t \le t_f, \\ u(0.1, t) &= e^{-0.1-t} + (0.001)t + (0.1)t^3 + t^2 + 0.01, \quad 0 \le t \le t_f. \end{aligned}$$

The exact solution of this problem is

$$u(x, t) = e^{-x-t} + x^3 t + x t^3 + t^2 + x^2.$$

Table 4 The comparison between exact and Tikhonov solutions of $p(t)$ with noisy data

t	Exact	0th order Tikhonov	1st order Tikhonov	2nd order Tikhonov
0.01	0.990150	0.990149	0.990150	0.990150
0.02	0.980599	0.980596	0.980599	0.980598
0.1	0.914837	0.914780	0.914836	0.914825
0.11	0.907934	0.907865	0.907932	0.907918
0.5	0.856531	0.855188	0.856137	0.856094
0.51	0.860596	0.859197	0.860186	0.860145
0.8	1.089329	1.085894	1.088756	1.088546
0.81	1.100958	1.097442	1.100387	1.100166
0.9	1.216570	1.212314	1.216003	1.215702
0.91	1.230624	1.226286	1.230056	1.229749
1	1.367879	1.362790	1.367285	1.366945
	S	2.402e−003	4.086e−004	5.448e−004

Table 5 The comparison between exact and Tikhonov solutions of $u(0.2, t)$ with noisy data

t	Exact	0th order Tikhonov	1st order Tikhonov	2nd order Tikhonov
0.01	0.850764	0.850765	0.850765	0.850765
0.02	0.843080	0.843082	0.843081	0.843081
0.1	0.791818	0.791850	0.791824	0.791831
0.11	0.786693	0.786731	0.786701	0.786708
0.5	0.815585	0.816291	0.815888	0.815874
0.51	0.822354	0.823091	0.822669	0.822650
0.8	1.156679	1.158589	1.157098	1.157077
0.81	1.173087	1.175043	1.173501	1.173487
0.9	1.335871	1.338252	1.336227	1.336287
0.91	1.355653	1.358082	1.356002	1.356071
1	1.549194	1.552046	1.549486	1.549625
	S	1.332e−003	2.907e−004	2.910e−004

Our results obtained for $u(0, t)$, $u(0.2, t)$ and $u(x, t)$ when $t_f = 1$, $\Delta t = 0.01$ and $\Delta x = \frac{1}{4}$ with noisy data (noisy data = input data + (0.01)rand(1)) are presented in Tables 4, 5 and 6 and Figs. 10, 11, 12, 13, 14, 15, 16, 17 and 18.

4 Conclusion

A numerical method, to estimate unknown boundary condition is proposed for two types of the inverse problems, the heat problem (2.6) and the wave problem (2.19), by using Haar basis method. The following results are obtained.

1. The present study, successfully applies the numerical method to inverse problems.

Table 6 The comparison between exact and Tikhonov solutions of $u(x, 0.5)$ with noisy data

t	Exact	0th order Tikhonov	1st order Tikhonov	2nd order Tikhonov
0	0.856531	0.855211	0.856114	0.856076
0.1	0.821812	0.821812	0.821812	0.821812
0.2	0.815585	0.816278	0.815838	0.815847
0.3	0.840329	0.841219	0.840752	0.840744
0.4	0.898570	0.899482	0.899091	0.899071
0.5	0.992879	0.993755	0.993490	0.993469
0.6	1.125871	1.126669	1.126523	1.126504
0.7	1.300194	1.300887	1.300791	1.300772
0.8	1.518532	1.519118	1.519017	1.518998
0.9	1.783597	1.783947	1.783873	1.783860
1	2.098130	2.098130	2.098130	2.098130
	S	7.696e−004	4.641e−004	4.545e−004

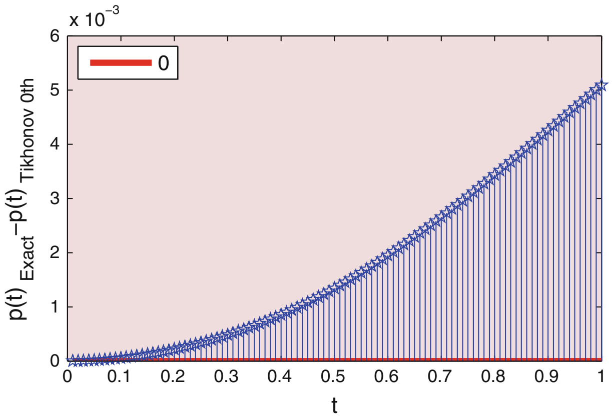

Fig. 10 Difference between the $p(t)_{\text{Exact}}$ and $p(t)_{\text{0th order Tikhonov}}$ of problem (2.19) with noisy data

2. Numerical results show that an excellent estimation can be obtained within a couple of minutes CPU time at pentium(R) 4 CPU 3.20 GHz.
3. The present method has been found stable with respect to small perturbation in the input data.
4. Numerical results show that our approximations of unknown function using the (1st and 2nd order) Tikhonov regularization combined with the Haar basis method, are more accurate than those obtained by the 0th order Tikhonov regularization with noisy data.

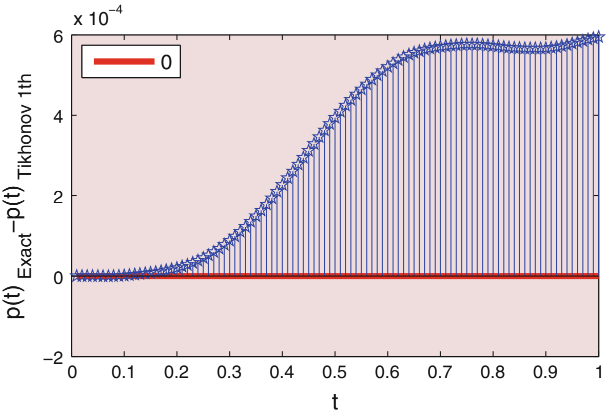

Fig. 11 Difference between the $p(t)_{\text{Exact}}$ and $p(t)_{\text{1st order Tikhonov}}$ of problem (2.19) with noisy data

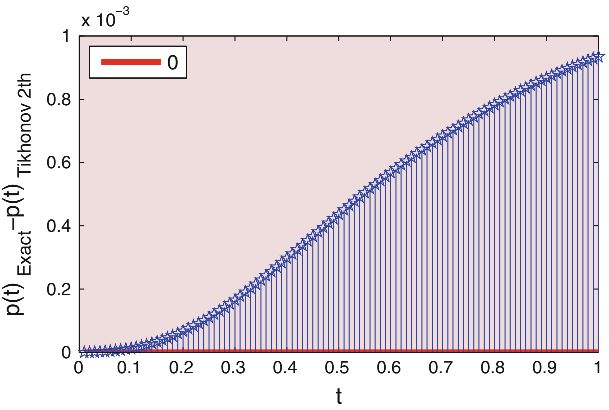

Fig. 12 Difference between the $p(t)_{\text{Exact}}$ and $p(t)_{\text{2nd order Tikhonov}}$ of problem (2.19) with noisy data

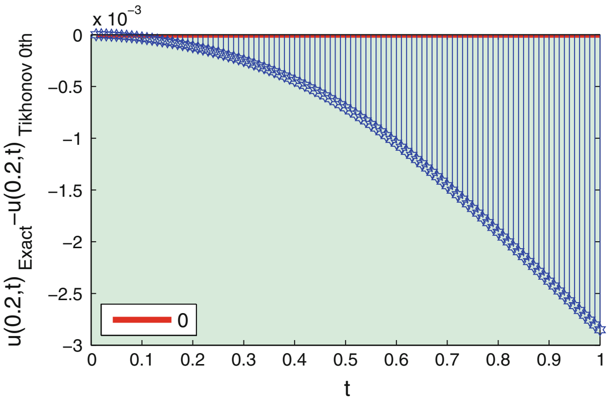

Fig. 13 Difference between the $u(0.2, t)_{\text{Exact}}$ and $u(0.2, t)_{\text{0th order Tikhonov}}$ of problem (2.19) with noisy data

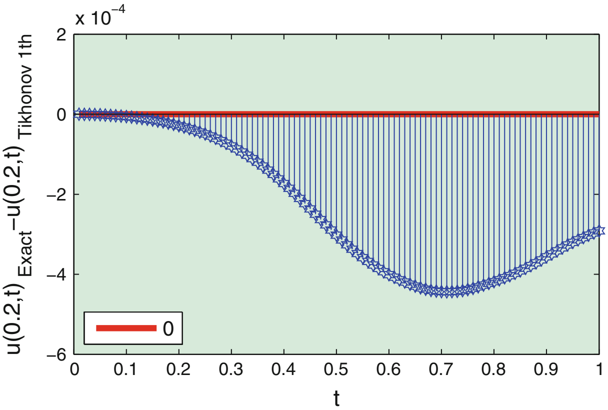

Fig. 14 Difference between the $u(0.2,t)_{\text{Exact}}$ and $u(0.2,t)_{\text{1st order Tikhonov}}$ of problem (2.19) with noisy data

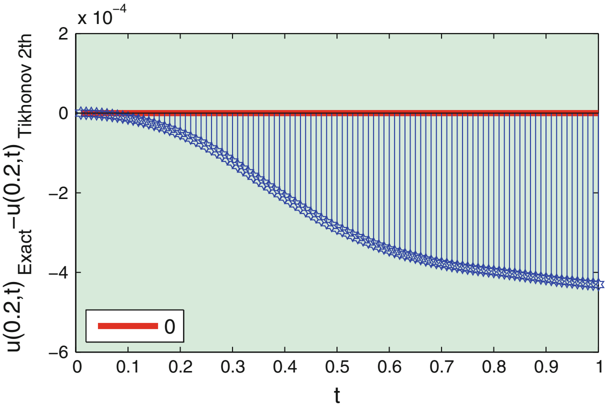

Fig. 15 Difference between the $u(0.2,t)_{\text{Exact}}$ and $u(0.2,t)_{\text{2nd order Tikhonov}}$ of problem (2.19) with noisy data

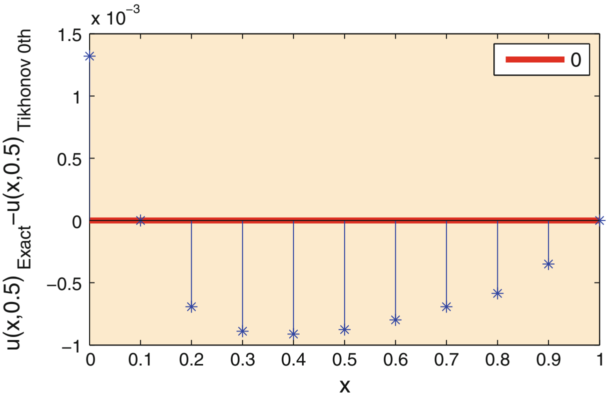

Fig. 16 Difference between the $u(x,0.5)_{\text{Exact}}$ and $u(x,0.5)_{\text{0th order Tikhonov}}$ of problem (2.19) with noisy data

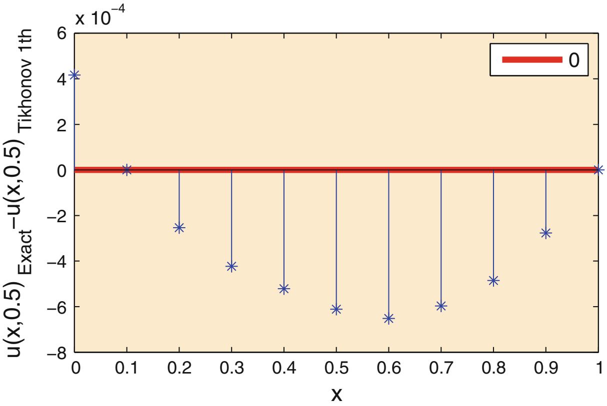

Fig. 17 Difference between the $u(x, 0.5)_{\text{Exact}}$ and $u(x, 0.5)_{\text{1st order Tikhonov}}$ of problem (2.19) with noisy data

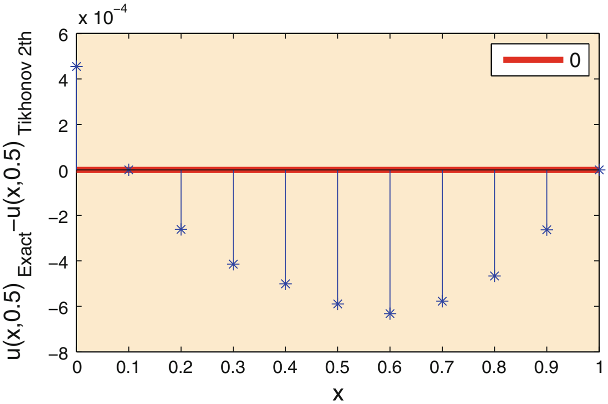

Fig. 18 Difference between the $u(x, 0.5)_{\text{Exact}}$ and $u(x, 0.5)_{\text{2nd orderTikhonov}}$ of problem (2.19) with noisy data

References

1. M. Abtahi, R. Pourgholi, A. Shidfar, Existence and uniqueness of solution for a two dimensional nonlinear inverse diffusion problem. Nonlinear Anal. Theory Methods Appl. **74**, 2462–2467 (2011)
2. O.M. Alifanov, *Inverse Heat Transfer Problems* (Springer, New York, 1994)
3. J.V. Beck, B. Blackwell, C.R.St. Clair, *Inverse Heat Conduction: Ill Posed Problems* (Wiley, New York, 1985)
4. J.V. Beck, B. Blackwell, A. Haji-sheikh, Comparison of some inverse heat conduction methods using experimental data. Int. J. Heat Mass Transf. **3**, 3649–3657 (1996)
5. J.V. Beck, D.C. Murio, Combined function specification-regularization procedure for solution of inverse heat condition problem. AIAA J. **24**, 180–185 (1986)
6. J.M.G. Cabeza, J.A.M. Garcia, A.C. Rodriguez, A sequential algorithm of inverse heat conduction problems using singular value decomposition. Int. J. Thermal Sci. **44**, 235–244 (2005)
7. C.F. Chen, C.H. Hsiao, Haar wavelet method for solving lumped and distributed-parameter systems. IEE Proc. Part D **144**(1), 87–94 (1997)
8. L. Elden, A note on the computation of the generalized cross-validation function for ill-conditioned least squares problems. BIT **24**, 467–472 (1984)

9. G.H. Golub, M. Heath, G. Wahba, Generalized cross-validation as a method for choosing a good ridge parameter. Technometrics **21**(2), 215–223 (1979)
10. P.C. Hansen, Analysis of discrete ill-posed problems by means of the L-curve. SIAM Rev. **34**, 561–580 (1992)
11. A. Haar, Zur theorie der orthogonalen Funktionsysteme. Math. Ann. **69**, 331–371 (1910)
12. G. Hariharan, K. Kannan, K.R. Sharma, Haar wavelet method for solving Fisher's equation. Appl. Math. Comput. **211**, 284–292 (2009)
13. C.H. Hsiao, W.J. Wang, Haar wavelet approach to nonlinear stiff systems. Math. Comput. Simul. **57**, 347–353 (2001)
14. C.-H. Huang, Y.-L. Tsai, A transient 3-D inverse problem in imaging the time- dependentlocal heat transfer coefficients for plate fin. Appl. Therm. Eng. **25**, 2478–2495 (2005)
15. C.-H. Huanga, C.-Y. Yeha, H.R.B. Orlande, A nonlinear inverse problem in simultaneously estimating the heat and mass production rates for a chemically reacting fluid. Chem. Eng. Sci. **58**(16), 3741–3752 (2003)
16. R. Kalpana, B.S. Raja, Haar wavelet method for the analysis of transistor circuits. Int. J. Electron. Commun. (AEU) **61**, 589–594 (2007)
17. C.L. Lawson, R.J. Hanson, *Solving Least Squares Problems* (SIAM, Philadelphia, 1995)
18. L. Martin, L. Elliott, P.J. Heggs, D.B. Ingham, D. Lesnic, X. Wen, Dual reciprocity boundary element method solution of the cauchy problem for Helmholtz-type equations with variable coefficients. J. Sound Vib. **297**, 89–105 (2006)
19. H. Molhem, R. Pourgholi, A numerical algorithm for solving a one-dimensional inverse heat conduction problem. J. Math. Stat. **4**(1), 60–63 (2008)
20. D.A. Murio, *The Mollification Method and the Numerical Solution of Ill-Posed Problems* (Wiley, New York, 1993)
21. D.C. Murio, J.R. Paloschi, Combined mollification-future temperature procedure for solution of inverse heat conduction problem. J. Comput. Appl. Math. **23**, 235–244 (1988)
22. R. Pourgholi, N. Azizi, Y.S. Gasimov, F. Aliev, H.K. Khalafi, Removal of numerical instability in the solution of an inverse heat conduction problem. Commun. Nonlinear Sci. Numer. Simul. **14**(6), 2664–2669 (2009)
23. R. Pourgholi, M. Rostamian, A numerical technique for solving IHCPs using Tikhonov regularization method. Appl. Math. Model. **34**(8), 2102–2110 (2010)
24. R. Pourgholi, M. Rostamian, M. Emamjome, A numerical method for solving a nonlinear inverse parabolic problem. Inverse Probl. Sci. Eng. **18**(8), 1151–1164 (2010)
25. K.K. Sun, B.S Jung, W.L. Lee, An inverse estimation of surface temperature using the maximum entropy method. Int. Commun. Heat Mass Transf. **34**, 37–44 (2007)
26. M. Tadi, Inverse heat conduction based on boundary measurement. Inverse Probl. **13**, 1585–1605 (1997)
27. A.N. Tikhonov, V.Y. Arsenin, *On the Solution of Ill-Posed Problems* (Wiley, New York, 1977)
28. A.N. Tikhonov, V.Y. Arsenin, *Solution of Ill-Posed Problems* (V.H. Winston and Sons, Washington, 1977)
29. G. Wahba, *Spline Models for Observational Data, CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 59* (SIAM, Philadelphia, 1990)
30. J. Zhou, Y. Zhang, J.K. Chen, Z.C. Feng, Inverse heat conduction in a composite slab With pyrolysis effect and temperature-dependent thermophysical properties. J. Heat Transf. **132**(3), 034502 (2010)